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Christoffel-type functions for m-orthogonal polynomials for Freud weights☆

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Abstract

This paper gives upper and lower bounds of the Christoffel-type functions $\lambda_{jn}(W^m, m; x)$, $j = m-2, m-4, \dots, m-2[m/2]$, for the m-orthogonal polynomials for a Freud weight $W = e^{-Q}$, which are given as follows. Let $a_n = a_n(Q)$ be the nth Mhaskar–Rahmanov–Saff number, $\phi_n(x) = \max\{n^{-2/3}, 1-|x|/a_n\}$, and $d > 0$. Assume that $Q \in C(\mathbf{R})$ is even, $Q'' \in C[0,\infty)$, $Q'(x) > 0$, $x \in (0,\infty)$, $Q(0) = 0$, and for some $A, B > 1$

$$A \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant B, \quad x \in (0,\infty).$$

Then for $x \in \mathbf{R}$

$$\lambda_{jn}(W^m, m; x) \geqslant \begin{cases} c\left(\dfrac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{-1/2}, & m \text{ is even}, \quad j = 0, \\[2ex] c\left(\dfrac{a_n}{n}\right)^{j+1} W(x)^m & \text{otherwise}, \end{cases}$$

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and for $|x| \leqslant a_n(1+dn^{-2/3})$

$$\lambda_{jn}(W^m, m; x) \leqslant c\left(\frac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{(1-m)/2}.$$

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1. Introduction and main results

We denote by $\mathbf{N}$, $\mathbf{N}_1$, or $\mathbf{N}_2$ the set of positive, odd, and even integers, respectively. We also denote by $\mathbf{R}$ the set of real numbers.

Let μ be a nondecreasing function on $\mathbf{R}$ with infinitely many points of increase such that all moments of $d\mu$ are finite. We call $d\mu$ a measure. If μ happens to be absolutely continuous then we will usually write w instead of μ' and will call w a weight. The symbol $\mathbf{P}_n$ stands for the set of algebraic polynomials of degree at most n. The symbol ∂P denotes the exact degree of the polynomial $P \neq 0$, i.e., $P \in \mathbf{P}_{\partial P} \setminus \mathbf{P}_{\partial P-1}$.

We denote by $c, c_1, \ldots$ positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. We write $a_n \sim b_n$ if $c_1 \leqslant a_n/b_n \leqslant c_2$ holds for every n. The notations $a(x) \sim b(x)$ and $a_n(x) \sim b_n(x)$ have similar meaning.

Throughout this paper let $m \in \mathbf{N}$ $(m \geqslant 2)$, $\mathbf{M}_1 = \{j \leqslant m-3 : m-j \in \mathbf{N}_1\}$, and $\mathbf{M}_2 = \{j \leqslant m-2 : m-j \in \mathbf{N}_2\}$. Put $\mathbf{P}_n^* = \{P(x) = c(x-y_1)\cdots(x-y_r) :\ c, y_1, \ldots, y_r \in \mathbf{R},\ r \leqslant n\}$ and $\mathbf{P}_n^*(x) = \{P \in \mathbf{P}_n^* : P(x) = 1\}$ for $x \in \mathbf{R}$. We agree $\mathbf{P}_0^* = \mathbf{P}_0$.

We define the m-monic orthogonal polynomials

$$P_n(d\mu, m; x) = x^n + \cdots, \quad n = 0, 1, \ldots,$$

for which

$$\int_{\mathbf{R}} |P_n(d\mu, m; x)|^m \, d\mu(x) = \min_{P(x)=x^n+\cdots} \int_{\mathbf{R}} |P(x)|^m \, d\mu(x). \tag{1.1}$$

According to Theorem 4 in [1], if $x_k = x_{kn}(d\mu, m)$ with

$$x_1 < x_2 < \cdots < x_n \tag{1.2}$$

are the zeros of $P_n(d\mu, m; x)$ then the Gaussian quadrature formula

$$\int_{\mathbf{R}} f(x) \operatorname{sgn} P_n(d\mu, m; x)^m \, d\mu(x) = \sum_{k=1}^{n} \sum_{j=0}^{m-2} \lambda_{kj} f^{(j)}(x_k) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_{mn-1}$, where the *Christoffel numbers* $\lambda_{kj} = \lambda_{kjn}(d\mu, m)$ are given by

$$\lambda_{kj} = \int_{\mathbf{R}} A_{kj}(x) \operatorname{sgn} P_n(d\mu, m; x)^m \, d\mu(x) \tag{1.4}$$

and $A_{kj} \in \mathbf{P}_{mn-1}$ are the fundamental polynomials of Hermite interpolation, which satisfy

$$A_{kj}^{(p)}(x_q) = \delta_{kq}\delta_{jp}, \quad j, p = 0, 1, \ldots, m-1, \ \ k, q = 1, 2, \ldots, n.$$

As we know, orthogonal polynomials ($m = 2$) have a long history of study and a classical theory. One of the important contents of this theory are the *Christoffel functions*

$$\lambda_n(d\mu; x) = \min_{P \in \mathbf{P}_{n-1}, P(x)=1} \int_{\mathbf{R}} P(t)^2 d\mu(t), \tag{1.5}$$

which are closely related to the *Christoffel numbers*

$$\lambda_{kn}(d\mu) = \lambda_n(d\mu; x_{kn}(d\mu)), \quad k = 1, 2, \ldots, n.$$

Here we accept the notation $P_n(d\mu) = P_n(d\mu, 2)$, $x_{kn}(d\mu) = x_{kn}(d\mu, 2)$, etc. The study and applications of the Christoffel functions can be found in [7].

The author in [8,11] defines the Christoffel-type functions $\lambda_{jn}(d\mu, m; x)$, which are the extension of $\lambda_n(d\mu; x)$ to the m-orthogonal polynomials and are given as follows.

Given a fixed point $x \in \mathbf{R}$, an index j, $0 \leqslant j \leqslant m-2$, and $n \in \mathbf{N}$, for $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ let the polynomial

$$A_j(P, x; t) = A_{jnm}(P, x; t) = \frac{1}{j!}(t-x)^j B_j(P, x; t) P(t)^m \tag{1.6}$$

with $B_j(P, x; \cdot) \in \mathbf{P}_{m-j-2}$ satisfy the conditions

$$A_j^{(i)}(P, x; x) = \delta_{ij}, \quad i = 0, 1, \ldots, m-2. \tag{1.7}$$

It is easy to see that $A_j(P, x; t)$ must exist and be unique.

Definition 1.1 (*Shi [11, Definition 1.1]*). The Christoffel-type function $\lambda_{jn}(d\mu, m; x)$ with respect to $d\mu$ is defined by

$$\lambda_{jn}(d\mu, m; x) = \inf_{P \in \mathbf{P}_{n-1}^*(x)} \int_{\mathbf{R}} A_j(P, x; t) \mathrm{sgn}[(t-x)P(t)]^m d\mu(t) \tag{1.8}$$

for $j \in \mathbf{M}_2$ and by

$$\lambda_{jn}(d\mu, m; x) = \int_{\mathbf{R}} A_j(P, x; t) \mathrm{sgn}[(t-x)P(t)]^m d\mu(t) \tag{1.9}$$

for $j \in \mathbf{M}_1$, where the polynomial P in (1.9) is the solution of (1.8) in the case when $j \in \mathbf{M}_2$.

According to Theorem 2.1 in [11] there is a unique polynomial $P \in \mathbf{P}_{n-1}^*(x)$ such that Eq. (1.8) holds for *every* $j \in \mathbf{M}_2$. So the definition of $\lambda_{jn}(d\mu, m; x)$ for $j \in \mathbf{M}_1$ is reasonable. Meanwhile we have

$$\lambda_{0n}(d\mu, 2; x) = \lambda_n(d\mu; x)$$

by Corollary 2.2 in [11] and

$$\lambda_{kjn}(d\mu, m) = [\mathrm{sgn}\, P_n'(d\mu, m; x_{kn}(d\mu, m))^m]\lambda_{jn}(d\mu, m; x_{kn}(d\mu, m)), \quad k = 1, 2, \ldots, n$$

by Theorem 2.3 in [11].

In [9,11] the author gives estimations of $\lambda_{jn}(u, m; x)$ for a weight u satisfying

$$u \sim w \quad \text{a.e.,} \tag{1.10}$$

where w is a *generalized Jacobi weight*:

$$w(x) = \prod_{i=1}^{r} |x - t_i|^{p_i}, \quad |x| < 1, \qquad w(x) = 0, \quad |x| \geqslant 1,$$
$$-1 = t_1 < t_2 < \cdots < t_r = 1 \quad (r \geqslant 2), \qquad p_i > -1, \quad i = 1, 2, \ldots, r. \tag{1.11}$$

Theorem 1.1 (*Shi [11, Theorem 3.3]*). *Let relation* (1.10) *prevail. Then with the constants associated with the symbol* $\sim$ *depending on u and m*,

$$\lambda_{jn}(u, m; x) \sim \lambda_n(u; x)\Delta_n(x)^j \sim \frac{1}{n} w_n(x)\Delta_n(x)^j, \qquad x \in [-1, 1], \quad j \in \mathbf{M}_2. \tag{1.12}$$

Here

$$w_n(x) = \left[(1+x)^{1/2} + \frac{1}{n}\right]^{2p_1+1} \left[(1-x)^{1/2} + \frac{1}{n}\right]^{2p_r+1} \prod_{i=2}^{r-1} \left[|x - t_i| + \frac{1}{n}\right]^{p_i} \tag{1.13}$$

and

$$\Delta_n(x) = \frac{(1-x^2)^{1/2}}{n} + \frac{1}{n^2}.$$

Freud weights on an infinite interval are as significant as generalized Jacobi weights on a finite interval.

Definition 1.2 (*Lubinsky and Mastroianni [5, Definition 1.1]*). Let $W = e^{-Q}$ where $Q \in C(\mathbf{R})$ is even, $Q'' \in C(0, \infty)$, $Q'(x) > 0$, $x \in (0, \infty)$, and for some $A, B > 1$

$$A \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant B, \qquad x \in (0, \infty). \tag{1.14}$$

Then we write $W \in \mathcal{F}$.

Assume, further, that $Q(0) = 0$ and $Q' \in C[0, \infty)$. In this case we write $W \in \mathcal{F}^*$.

For $W \in \mathcal{F}$ the qth *Mhaskar–Rahmanov–Saff number* $a_q = a_q(Q)$ is defined by the positive root of the equation

$$q = \frac{2}{\pi} \int_0^1 a_q t Q'(a_q t)(1 - t^2)^{-1/2} dt, \qquad q > 0. \tag{1.15}$$

Levin and Lubinsky give the following important results, in which

$$\phi_n(x) = \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}.$$

Theorem 1.2 (*Levin and Lubinsky [4, Theorem 1.1]*). *Let $W \in \mathcal{F}$, $n \in \mathbf{N}$, and $d > 0$. Then*

$$\begin{cases} \lambda_n(W^2; x) \sim \dfrac{a_n}{n} W(x)^2 \phi_n(x)^{-1/2}, & |x| \leqslant a_n(1 + dn^{-2/3}), \\ \lambda_n(W^2; x) \geqslant c \dfrac{a_n}{n} W(x)^2 \phi_n(x)^{-1/2}, & x \in \mathbf{R}. \end{cases} \tag{1.16}$$

In this paper we shall give upper and lower bounds of the Christoffel-type functions $\lambda_{jn}(W^m, m; x)$ with $j \in \mathbf{M}_2$ for a Freud weight W.

Theorem 1.3. *Let $W \in \mathcal{F}^*$, $n \in \mathbf{N}$, $d > 0$, and $j \in \mathbf{M}_2$. Then for $x \in \mathbf{R}$*

$$\lambda_{jn}(W^m, m; x) \geqslant \begin{cases} c\left(\dfrac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{-1/2}, & m \in \mathbf{N}_2,\ j = 0, \\ c\left(\dfrac{a_n}{n}\right)^{j+1} W(x)^m & otherwise, \end{cases} \tag{1.17}$$

and for $|x| \leqslant a_n(1 + dn^{-2/3})$

$$\lambda_{jn}(W^m, m; x) \leqslant c\left(\frac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{(1-m)/2}. \tag{1.18}$$

We shall give some auxiliary lemmas in Section 2 and the proof of Theorem 1.3 in Section 3.

2. Auxiliary lemmas

We need some known results. Here $\|f\| = \sup_{x \in \mathbf{R}} |f(x)|$, $f \in C(\mathbf{R})$.

Lemma 2.1 (*Shi [11, Lemma 2.1]*). *We have*

$$B_j(P, x; t) = \sum_{i=0}^{m-j-2} b_i (t - x)^i, \tag{2.1}$$

where

$$b_i = b_i(P, x) = \frac{1}{i!}\left[P(t)^{-m}\right]_{t=x}^{(i)}, \quad i = 0, 1, \ldots . \tag{2.2}$$

Moreover, for $P \in \mathbf{P}^*_{n-1}(x)$ and $j \in \mathbf{M}_2$

$$b_{m-j-2} > 0, \quad B_j(P, x; t) > 0, \quad t \in \mathbf{R}. \tag{2.3}$$

Lemma 2.2 (*Shi [8, Theorem 3]*). *Let $m \in \mathbf{N}_2$. We have*

$$\lambda_{0n}(d\mu, m; x) \geqslant \lambda_{mn/2}(d\mu; x). \tag{2.4}$$

Lemma 2.3 (*Nevai [6, Lemma 6.3.8, p. 108]*). *Let $v(x) = (1 - x^2)^{-1/2}$ and*

$$K_n(v; x, t) = \frac{T_n(x)T_{n-1}(t) - T_{n-1}(x)T_n(t)}{\pi(x - t)}, \quad n \geqslant 2,$$

where T_n stands for the nth Chebyshev polynomial of the first kind. Then

$$|K_n(v; x, t)| \leqslant c \min \left\{ n, \frac{(1-x^2)^{1/2} + (1-t^2)^{1/2}}{|x-t|} \right\}, \quad x, t \in [-1, 1], \tag{2.5}$$

where c is an absolute constant.

Lemma 2.4 (*Freud [2, (3.7), p. 102; p. 104]*). *Let*

$$\pi_{n-1}(x, t) = \frac{K_n(v; x, t)}{K_n(v; x, x)}.$$

Then

$$|\pi_{n-1}(x, t)| \leqslant 4, \quad n \geqslant 3, \quad x, t \in [-1, 1] \tag{2.6}$$

and

$$K_n(v; x, x) \sim n, \quad |x| \leqslant 1. \tag{2.7}$$

By [2, Theorem 3.1, p. 19] the polynomial $\pi_{n-1}(x, t)$ in t has simple real zeros only and hence

$$\pi_{n-1}(x, \cdot) \in \mathbf{P}^*_{n-1}(x). \tag{2.8}$$

Lemma 2.5 (*Hardy [3, Theorem 27, pp. 71–72]*). *Let $A, B, p \geqslant 0$ and $AB + p > 0$. Then*

$$(A + B)^p \leqslant c(p)(A^p + B^p). \tag{2.9}$$

Lemma 2.6. *Let $\ell \in \mathbf{N}$, $s = 1 + [n/\ell]$, $a > 0$, and*

$$S_{n,a,x}(t) = \pi_{s-1}(x/a, t/a)^\ell. \tag{2.10}$$

Then

$$|b_i(S_{n,a,x}, x)| \leqslant c(\ell, i, m)\, [a\Delta_n(x/a)]^{-i}, \quad |x| \leqslant a. \tag{2.11}$$

Proof. By Bernstein inequality and (2.6) for $|x|, |t| \leqslant a$

$$\begin{aligned} &|[S_{n,a,x}(t)^m]^{(i)}_{t=x}| \\ &\quad = \left| \frac{d^i}{dt^i} \pi_{s-1}(x/a, t/a)^{\ell m} \Big|_{t=x} \right| \\ &\quad \leqslant c[a\Delta_n(x/a)]^{-i} \max_{|x|,|t| \leqslant a} |\pi_{s-1}(x/a, t/a)^{\ell m}| \\ &\quad \leqslant c[a\Delta_n(x/a)]^{-i}. \end{aligned}$$

Since

$$\left[S_{n,a,x}(t)^m S_{n,a,x}(t)^{-m} \right]^{(i)} = 0, \quad i \geqslant 1,$$

we have

$$b_i(S_{n,a,x},x) = -\sum_{v=0}^{i-1} b_v(S_{n,a,x},x)\left\{\frac{1}{(i-v)!}[S_{n,a,x}(t)^m]_{t=x}^{(i-v)}\right\}$$

and hence

$$\begin{aligned}|b_i(S_{n,a,x},x)| &\leqslant \sum_{v=0}^{i-1} |b_v(S_{n,a,x},x)|\,|[S_{n,a,x}(t)^m]_{t=x}^{(i-v)}|\\ &\leqslant c\sum_{v=0}^{i-1}[a\Delta_n(x/a)]^{v-i}|b_v(S_{n,a,x},x)|.\end{aligned}$$

By induction we obtain (2.11). □

Lemma 2.7 (*Shi [10, Lemma 1]*). *If* $u(t)\leqslant Cw(t)$ *and*

$$N\geqslant 8+2\left[p_1+p_r+\frac{1}{2}\sum_{i=2}^{r-1}|p_i|\right], \tag{2.12}$$

where w *is given in* (1.11) *and the symbol* $[y]$ *denotes the integral part of the real number* y, *then the inequality*

$$S=\int_{-1}^{1}|\pi_{n-1}(x,t)|^N u(t)dt\leqslant c(C,N,w)n^{-1}w_n(x) \tag{2.13}$$

holds for all $x\in[-1,1]$ *and* $n\in\mathbf{N}$.

Lemma 2.8. *Let* $u(t)\leqslant Cw(t),\ 0\leqslant p\leqslant m$,

$$N=4m+m\left[p_1+p_r+\frac{1}{2}\sum_{i=2}^{r-1}|p_i|\right],\quad s=1+\left[\frac{m(n-1)}{N+m}\right], \tag{2.14}$$

and

$$P_{n-1}(t)=\pi_{s-1}(x,t)^{(N+m)/m}, \tag{2.15}$$

where w *is given in* (1.11). *Then the inequality*

$$S=\int_{-1}^{1}|P_{n-1}(t)|^m|t-x|^p u(t)dt\leqslant c(C,m,w)\lambda_n(w;x)\Delta_n(x)^p \tag{2.16}$$

holds for all $x\in[-1,1]$ *and* $n\in\mathbf{N}$.

Proof. By Lemma 2.7 for $0\leqslant q\leqslant m$

$$\int_{-1}^{1}|\pi_{s-1}(x,t)|^N(1-t^2)^q u(t)dt\leqslant cs^{-1}\delta_s(x)^q w_s(x),\quad |x|\leqslant 1,\quad s\in\mathbf{N}, \tag{2.17}$$

where $\delta_s(x) = (1-x^2)^{1/2} + s^{-1}$. Since $\partial P_{n-1} = (s-1)(N+m)/m \leqslant n-1$, by (2.5)–(2.7), (2.9), and (2.17) we have

$$\begin{aligned}
S &= \int_{-1}^{1} |\pi_{s-1}(x,t)|^{N+m}|t-x|^p u(t)dt \\
&= \int_{-1}^{1} |\pi_{s-1}(x,t)|^{N+m-p}|(t-x)K_s(v;x,t)|^p K_s(v;x,x)^{-p}u(t)dt \\
&\leqslant cs^{-p}\int_{-1}^{1} |\pi_{s-1}(x,t)|^N[(1-x^2)^{1/2}+(1-t^2)^{1/2}]^p u(t)dt \\
&\leqslant cs^{-p}\left[(1-x^2)^{p/2}\int_{-1}^{1}|\pi_{s-1}(x,t)|^N u(t)dt + \int_{-1}^{1}|\pi_{s-1}(x,t)|^N(1-t^2)^{p/2}u(t)dt\right] \\
&\leqslant cs^{-p-1}w_s(x)[(1-x^2)^{p/2}+\delta_s(x)^p] \\
&\leqslant cs^{-p-1}\delta_s(x)^p w_s(x) \\
&= cs^{-1}w_s(x)\Delta_s(x)^p \\
&\leqslant cn^{-1}w_n(x)\Delta_n(x)^p \\
&\leqslant c\lambda_n(w;x)\Delta_n(x)^p. \qquad \square
\end{aligned}$$

Lemma 2.9 (*Levin and Lubinsky [4, Lemma 5.2]*). *Let $W \in \mathcal{F}$ and $d > 1$. Then the relation*

$$\left|\frac{a_p}{a_q} - 1\right| \sim \left|\frac{p}{q} - 1\right| \tag{2.18}$$

uniformly holds for $q \in (0,\infty)$ and $p \in [q/d, dq]$.

Lemma 2.10 (*Levin and Lubinsky [4, Theorem 1.8]*). *Let $W \in \mathcal{F}$, $0 < p \leqslant \infty$, and $d > 0$. Then for $P \in \mathbf{P}_n$*

$$\|PW\|_{L_p(\mathbf{R})} \leqslant c\|PW\|_{L_p(|x|\leqslant a_n(1-dn^{-2/3}))}. \tag{2.19}$$

Lemma 2.11 (*Levin and Lubinsky [4, Theorem 1.9]*). *Let $W \in \mathcal{F}^*$. Then for $P \in \mathbf{P}_n$*

$$\|P'W\| \leqslant c\frac{n}{a_n}\|PW\|. \tag{2.20}$$

Lemma 2.12. *Let $W \in \mathcal{F}$ and $q, \lambda > 0$. Then*

$$a_q(\lambda Q) = a_{q/\lambda}. \tag{2.21}$$

Proof. Replacing q by q/λ or Q by λQ in (1.15), we have

$$q = \frac{2}{\pi}\int_0^1 a_{q/\lambda}(Q)t(\lambda Q)'(a_{q/\lambda}(Q)t)(1-t^2)^{-1/2}dt$$

or

$$q = \frac{2}{\pi}\int_0^1 a_q(\lambda Q)t(\lambda Q)'(a_q(\lambda Q)t)(1-t^2)^{-1/2}dt,$$

respectively. Comparing these two equations and observing $a_{q/\lambda} = a_{q/\lambda}(Q)$, we obtain (2.21). $\square$

Following Levin and Lubinsky [4, pp. 485–488] define for $x \in [-1, 1]\backslash\{0\}$

$$\mu_n(x) = \frac{2}{\pi^2}\int_0^1 \frac{(1-x^2)^{1/2}}{(1-t^2)^{1/2}} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} dt.$$

Then by Lemma 7.1 in [4]

$$\mu_n(x) > 0, \quad x \in (-1, 1)\backslash\{0\} \tag{2.22}$$

and

$$\int_{-1}^{1} \mu_n(x) dx = 1. \tag{2.23}$$

Now first we choose

$$-1 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$$

so that

$$\int_{t_k}^{t_{k+1}} \mu_n(t) dt = \begin{cases} \dfrac{1}{2n}, & k = 0, n, \\ \\ \dfrac{1}{n}, & 1 \leqslant k \leqslant n-1. \end{cases} \tag{2.24}$$

Next, for a fixed $x \in \mathbf{R}$ we choose $\sigma_k = \sigma_{kn}(x)$, $1 \leqslant k \leqslant n$, as follows. For $t_i \leqslant x < t_{i+1}$ with $2 \leqslant i \leqslant n-2$, put

$$\sigma_k = \begin{cases} 0, & k = i, i+1, \\ 2, & k = i-1, i+2, \\ 1 & \text{otherwise}, \end{cases}$$

for $x \geqslant t_{n-1}$, put

$$\sigma_k = \begin{cases} 0, & k = n-1, n, \\ 3, & k = n-2, \\ 1 & \text{otherwise}, \end{cases}$$

for $x < t_2$, put

$$\sigma_k = \begin{cases} 0, & k = 1, 2, \\ 3, & k = 3, \\ 1 & \text{otherwise}. \end{cases}$$

Then we write

$$P_{n,x}(t) = \prod_{k=1}^{n} (t - t_k)^{\sigma_k} \tag{2.25}$$

and

$$R_{n,x}(t) = \frac{P_{n,x}(t/a_n)}{P_{n,x}(x/a_n)}. \tag{2.26}$$

Lemma 2.13. *Let $d > 0$. If $W \in \mathcal{F}$ then*

$$\begin{aligned}\lambda_{n+1,\infty}(W; x) &:= \inf_{P\in\mathbf{P}_n,\, P(x)=1} \|PW\| \\ &\leqslant \|R_{n,x}W\| \leqslant cW(x), \quad |x| \leqslant a_n(1+dn^{-2/3}),\end{aligned} \tag{2.27}$$

and if $W \in \mathcal{F}^*$ then

$$|b_i(R_{n,x}, x)| \leqslant c(i, m, W)\left(\frac{n}{a_n}\right)^i, \quad |x| \leqslant a_n(1+dn^{-2/3}). \tag{2.28}$$

Proof. Inequality (2.27) is given in [4, pp. 514–515]. To prove (2.28) applying (2.20) repeatedly we obtain

$$\|[R^m_{n,x}]^{(i)}W^m\| \leqslant c\left(\frac{mn}{a_{mn}}\right)^i \|R^m_{n,x}W^m\| \leqslant c\left(\frac{n}{a_n}\right)^i \|R_{n,x}W\|^m. \tag{2.29}$$

Thus, by (2.29), (2.27), and (2.21) for $|x| \leqslant a_n(1+dn^{-2/3})$

$$\|[R^m_{n,x}]^{(i)}W^m\| \leqslant c\left(\frac{n}{a_n}\right)^i \|R_{n,x}W\|^m \leqslant c\left(\frac{n}{a_n}\right)^i W(x)^m$$

which implies

$$\left|[R_{n,x}(t)^m]^{(i)}_{t=x}\right| W(x)^m \leqslant c\left(\frac{n}{a_n}\right)^i W(x)^m$$

and hence

$$\left|[R_{n,x}(t)^m]^{(i)}_{t=x}\right| \leqslant c\left(\frac{n}{a_n}\right)^i.$$

By the same argument as that of Lemma 2.6 we obtain (2.28). □

3. Proof of theorem 1.3

The proof follows and properly modifies the ideas of Levin and Lubinsky in [4].

The first line of (1.17) follows from (2.4) and (1.16). Let us prove the second. Let $P \in \mathbf{P}^*_{n-1}(x)$ satisfy (1.8) with $d\mu(t) = W(t)^m dt$, that is,

$$\lambda_{jn}(W^m, m; x) = \int_{\mathbf{R}} |A_j(P, x; s)| W(s)^m ds.$$

Then by (1.5)

$$\begin{aligned}A_j(P, x; t)^2 &\leqslant \lambda_{mn}(W^{2m}; t)^{-1} \int_{\mathbf{R}} A_j(P, x; s)^2 W(s)^{2m} ds \\ &\leqslant \lambda_{mn}(W^{2m}; t)^{-1} \|A_j(P, x; \cdot)W^m\| \int_{\mathbf{R}} |A_j(P, x; s)| W(s)^m ds \\ &= \lambda_{mn}(W^{2m}; t)^{-1} \lambda_{jn}(W^m, m; x) \|A_j(P, x; \cdot)W^m\|.\end{aligned}$$

By (1.16) and (2.18)

$$[A_j(P, x; t)W(t)^m]^2 \leqslant c\frac{n}{a_n}\lambda_{jn}(W^m, m; x)\|A_j(P, x; \cdot)W^m\|$$

which yields

$$\|A_j(P,x;\cdot)W^m\|^2 \leqslant c\frac{n}{a_n}\lambda_{jn}(W^m,m;x)\|A_j(P,x;\cdot)W^m\|$$

and hence

$$\|A_j(P,x;\cdot)W^m\| \leqslant c\frac{n}{a_n}\lambda_{jn}(W^m,m;x). \tag{3.1}$$

Applying (2.20) j times and using (3.1), we obtain

$$W(x)^m = A_j^{(j)}(P,x;x)W(x)^m \leqslant c\left(\frac{n}{a_n}\right)^{j+1}\lambda_{jn}(W^m,m;x)$$

which gives the second line of (1.17).

Let us prove (1.18). We use (2.19) to get

$$\begin{aligned}\lambda_{jn}(W^m,m;x) &\leqslant \frac{1}{j!}\int_{\mathbf{R}}|(t-x)^j B_j(P,x;t)P(t)^m|W(t)^m\,dt\\ &\leqslant \frac{1}{j!}\sum_{i=0}^{m-2-j}|b_i(P,x)|\int_{\mathbf{R}}|t-x|^{i+j}|P(t)|^m W(t)^m\,dt\\ &\leqslant c\sum_{i=0}^{m-2-j}|b_i(P,x)|\int_{-a}^{a}|t-x|^{i+j}|P(t)|^m W(t)^m\,dt,\end{aligned} \tag{3.2}$$

where by (2.21) $a = a_{mn}(mQ)(1+dn^{-2/3}) = a_n(1+dn^{-2/3})$. Choose $P = P_{q-1}P_{n-q}$, where

$$P_{q-1}(t) = R_{q-1,x}(t) \tag{3.3}$$

and

$$P_{n-q}(t) = S_{n-q,a,x}(t) = \pi_{s-1}(x/a,t/a)^{(N+m)/m}, \quad s = 1+\left[\frac{m(n-q)}{N+m}\right]. \tag{3.4}$$

Then by (3.2)–(3.4)

$$\begin{aligned}&\lambda_{jn}(W^m,m;x)\\ &\quad\leqslant c\sum_{i=0}^{m-2-j}|b_i(P_{q-1}P_{n-q},x)|\,\|P_{q-1}^m W^m\|\int_{-a}^{a}|t-x|^{i+j}|P_{n-q}(t)|^m\,dt\\ &\quad= c\sum_{i=0}^{m-2-j}|b_i(P_{q-1}P_{n-q},x)|\,\|P_{q-1}^m W^m\|a^{i+j+1}\int_{-1}^{1}|t-x/a|^{i+j}|P_{n-q}(at)|^m\,dt\\ &\quad:= c\sum_{i=0}^{m-2-j}S_i.\end{aligned} \tag{3.5}$$

Let us estimate S_i for $0 \leqslant i \leqslant m-2-j$. By (2.27) and (2.21)

$$\|P_{q-1}^m W^m\| = \|P_{q-1}W\|^m \leqslant cW(x)^m, \quad |x| \leqslant a_{q-1}(1+d(q-1)^{-2/3}), \tag{3.6}$$

and applying Lemma 2.8 and Theorem 1.1

$$\int_{-1}^{1} |t - x/a|^{i+j} |P_{n-q}(at)|^m dt \leqslant c\Delta_{n-q}(x/a)^{i+j+1}, \quad |x| \leqslant a. \tag{3.7}$$

Also, by (2.28), (2.11), and (2.18)

$$\begin{aligned} &|b_i(P_{q-1}P_{n-q}, x)| \\ &\quad = \left| \frac{1}{i!} [P_{q-1}(t)^{-m} P_{n-q}(t)^{-m}]^{(i)}_{t=x} \right| \\ &\quad \leqslant c \sum_{v=0}^{i} |b_v(P_{q-1}, x)| \cdot |b_{i-v}(P_{n-q}, x)| \\ &\quad \leqslant c \sum_{v=0}^{i} \left(\frac{q-1}{a_{q-1}} \right)^v \left[a\Delta_{n-q}\left(\frac{x}{a}\right) \right]^{v-i} \\ &\quad \leqslant c \left[a_n\Delta_{n-q}\left(\frac{x}{a}\right) \right]^{-i} \sum_{v=0}^{i} \left[n\Delta_{n-q}\left(\frac{x}{a}\right) \right]^v, \\ &\qquad q \geqslant [n/2],\ |x| \leqslant a_{q-1}(1 + d(q-1)^{-2/3}). \end{aligned} \tag{3.8}$$

Then by (3.6)–(3.8) we have

$$\begin{aligned} S_i &\leqslant cW(x)^m \left[a_n\Delta_{n-q}\left(\frac{x}{a}\right) \right]^{j+1} \sum_{v=0}^{i} \left[n\Delta_{n-q}\left(\frac{x}{a}\right) \right]^v, \\ &\quad q \geqslant [n/2],\ |x| \leqslant a_{q-1}(1 + d(q-1)^{-2/3}). \end{aligned} \tag{3.9}$$

To estimate $\Delta_{n-q}(x/a)$ we separate three cases according to range of x.

Case 1: $|x| \leqslant a_{n/2}$. In this case let $q = [n/2]$. Then by (2.18)

$$\Delta_{n-q}\left(\frac{x}{a}\right) \leqslant \frac{c}{n}\phi_n(x)^{-1/2}. \tag{3.10}$$

Case 2: $a_{n/2} < |x| \leqslant a_{n(1-2n^{-2/3})}$. In this case let q satisfy

$$a_{q-1} < |x| \leqslant a_q.$$

Clearly, for n large enough we have

$$n/3 \leqslant q \leqslant n(1 - n^{-2/3}). \tag{3.11}$$

By (2.18)

$$1 - \frac{|x|}{a} \sim 1 - \frac{a_q}{a_n} \sim 1 - \frac{q}{n}$$

and hence by (3.11)

$$\frac{1}{(n-q)^2} \leqslant n^{-2/3} \leqslant 1 - \frac{q}{n} \sim 1 - \frac{|x|}{a}.$$

Thus

$$\Delta_{n-q}\left(\frac{x}{a}\right) \leqslant \frac{c}{n-q}\left(1-\frac{|x|}{a}\right)^{1/2}$$
$$= \frac{c}{n(1-q/n)}\left(1-\frac{|x|}{a}\right)^{1/2} \leqslant \frac{c}{n}\left(1-\frac{|x|}{a}\right)^{-1/2}$$

which implies (3.10).

Case 3: $a_{n(1-2n^{-2/3})} \leqslant |x| \leqslant a_n(1+dn^{-2/3})$. In this case let $q = n - [n^{1/3}]$. By (2.18)

$$1-\frac{|x|}{a} \leqslant cn^{-2/3} \sim \frac{1}{(n-q)^2}$$

and hence

$$\Delta_{n-q}\left(\frac{x}{a}\right) \leqslant \frac{c}{(n-q)^2} \sim n^{-2/3} \sim \frac{1}{n}\phi_n(x)^{-1/2}$$

which again implies (3.10). Also by (2.18)

$$\frac{|x|}{a_{q-1}} \leqslant \frac{a_n}{a_{q-1}} \leqslant 1 + c(q-1)^{-2/3}.$$

Using all these estimations inequality (3.9) with (3.10) yields

$$S_i \leqslant c\left(\frac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{-(i+j+1)/2}.$$

Then (1.18) follows from (3.5). □

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